MAXWELL STRESS AND ELECTROSTRICTION IN DIELECTRICS AND THEIR IMPLICATIONS FOR FRACTURE MECHANICS

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Abstract

In fracture mechanics of smart materials, the influence of electric fields on the propagation of cracks plays a key role. While the piezoelectric effect has been thoroughly investigated in this regard, nonlinear electrodynamic phenomena are oftentimes disregarded.

As an example, stemming from the microscopic LORENTZ force, electrostatic actions manifest themselves macroscopically in terms of surface tractions at discontinuities, body forces caused by graded fields and body couples due to local non-collinearity of electric field and polarization. All three of these manifestations are derived from the MAXWELL stress tensor, whose formulation in polarizable matter is still being debated to date [1]. By contrast, electrostriction represents a constitutive effect only inherent to dielectric materials, interlinking mechanical strains with the square of the electric field and polarization, respectively. Due to identical mathematical structures of electrostrictive and MAXWELL stresses in isotropic materials, both effects are sometimes treated equivalently.

In this work, these nonlinearities are studied with respect to an elliptic cavity in an infinite dielectric, providing a GRIFFITH crack in the limiting case of a vanishing semi-minor axis. In this context, predominant models of the MAXWELL stress tensor are compared and precisely distinguished from electrostriction, ultimately evaluating their individual contributions to crack tip loading and revealing a singularity differing from the well-known $1/\sqrt{r}$-type.

1. Point of departure and solution strategy

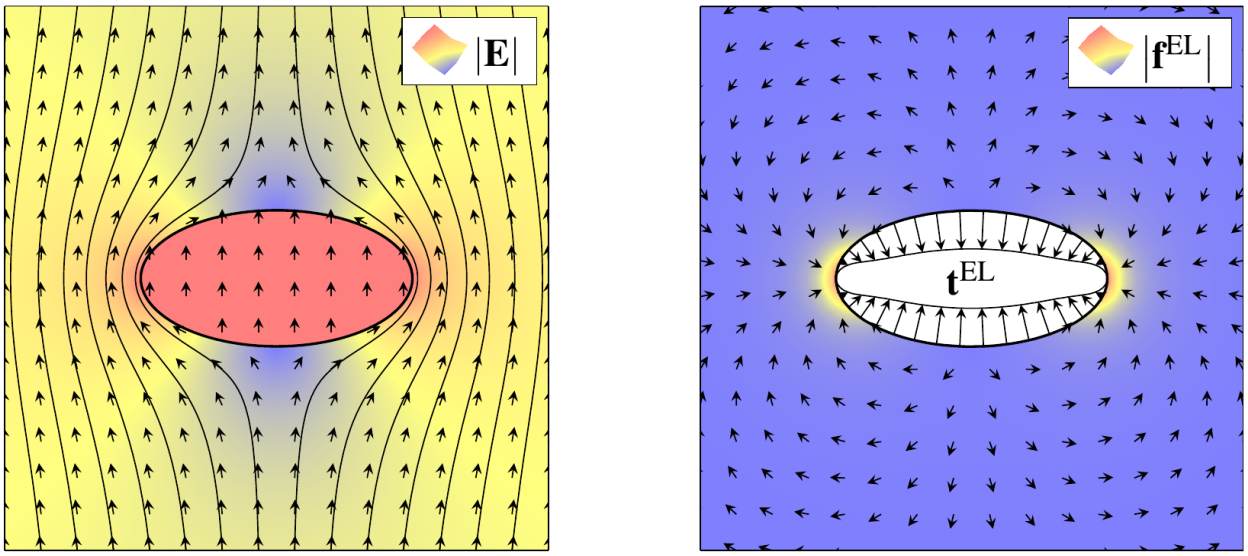
Sharp cracks represent the main subject of investigation in the field of linear elastic fracture mechanics, which has proven to be suitable for analysis of failure in multifunctional ceramics prone to brittle fracture. While usually assumed to be stress-free when dealing with solely mechanical loading scenarios, crack faces in dielectrics, on the contrary, are exposed to electrostatic surface forces naturally arising by virtue of different permittivities of the crack and the surrounding medium. Furthermore, due to gravitation having no impact on the crack tip loading, the effect of body forces on cracks has been scarcely explored. However, high or even singular gradients of the electric field prevailing near the crack tip induce significant electric body forces. Beyond that, anisotropy of material properties, mostly tracing back to the material's unit cell possessing a polar axis, does not only complicate the solution of boundary value problems, but also provokes electric body couples, which, to the best of the author's knowledge, have not yet been assessed in view of cracks, requiring the treatment of an asymmetric CAUCHY stress tensor within the theory of polar continua. Finally, the formulation of the MAXWELL stress tensor is commonly chosen from three established models according to either MINKOWSKI, EINSTEIN and LAUB or LORENTZ, each of them yielding unique electrostatic forces and couples.

Building on the pioneering works of SMITH and WARREN [2], who first solved the infinite electrostrictive dielectric plate with an elliptic cutout, the problem at hand is decoupled by neglecting the effect of deformation on permittivity, emanating from electrostrictive coupling, upon employment of infinitesimal strain theory, allowing for the successive treatment of electro- and elastostatics. Both subproblems are tackled within the framework of complex analysis, introducing holomorphic electric and mechanical potentials, whose complex derivatives provide the electric field and mechanical stresses intrinsically satisfying MAXWELL's equations and force equilibrium, respectively. Appropriate boundary and interface conditions at the ellipse as well as homogeneous conditions at infinity are formulated, the former of which, in order to facilitate the solution procedure, are mapped conformally onto a circle in an auxiliary

complex plane, where the potentials are obtained by means of CAUCHY's integral formulas in the style of MUSKHELISHVILI [3]. In the process, a widespread erroneous inverse function involved in the solution's transformation to the physical plane is rectified, enabling access to the field distributions in the entire plane.

2. Results

Fig. 1 illustrates the effect of a vertical electric field at infinity on a dielectric exhibiting an elliptic hole,



(a) **(b)**

Fig. 1 – Infinite dielectric with elliptic cavity subjected to remote vertical electric load, (a): field lines and unit vectors of the electric field $\mathbf{E}$, (b): unit vectors of the body force $\mathbf{f}^{EL}$ and surface tractions $\mathbf{t}^{EL}$ according to the EINSTEIN-LAUB approach.

whereupon the permittivity of the hole amounts to a thousandth of the bulk material's. As expected, the electric field circumvents the hole in (a), resulting in steep gradients of the electric field at the vertexes of the ellipse. Inside the flaw, the field intensity significantly exceeds the remote load, owing to the continuity of the normal component of the electric displacement at the elliptic interface. Emerging from this electric field distribution, surface tractions and body force according to EINSTEIN and LAUB are exemplarily depicted in (b). Apparently, body forces of significant magnitude are confined to narrow regions in the vicinity of the potential crack tips. In contrast, the surface tractions decay upon approaching the tips, overall acting as tensile load on the material's boundary.

References

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