Development and Application of the Hypercomplex Finite Element Method for Linear and Nonlinear Energy Release Rate Calculations

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Abstract

The augmentation of existing finite element codes to use complex and hypercomplex variables and algebras provides an accurate and straightforward method to compute the energy release rate (ERR) for linear and nonlinear solids. The basic concept is to introduce complex nodes defined by real and imaginary nodes. The real nodes define the geometry, as typically, and the imaginary nodes define the perturbation to the real mesh; For energy release rate computations, the crack is extended using imaginary coordinates surrounding the crack tip. The standard finite element formulation, e.g., same shape functions and quadrature rules, is then carried albeit with complex algebra. The solution of the complex system of equations then yields a complex displacement with the imaginary displacement equal to the derivative of the displacement with respect to the crack length. Subsequently, the energy release rate (the derivative of the strain energy with respect to the crack length) can be determined using from the complex strains and stresses. The results indicate that the ERR results are as accurate as the J integral but the method has several advantages: there are no contours to interrogate – only one result is provided, the method works for both linear and nonlinear materials with loading and unloading, unlike the J integral, and no integral formulation must be developed and implemented. Numerical examples demonstrate the accuracy of the method.

1. Introduction

The computation of the energy release rate is a fundamental component of a fracture control plan. A relatively new method for computing the ERR has been developed using a complex (or hypercomplex) finite element formulation called ZFEM[1-7]. The beauty of the ZFEM formulation and implementation is that the traditional FE formulation is used and *augmented* with imaginary coordinates and hypercomplex variables and algebra and minimal programming changes are required.

The implementation of a complex (or hypercomplex) differentiation method within a finite element formulation can be summarized in the following steps:

- Addition of imaginary degrees of freedom added through the inclusion of imaginary nodes (2n-1 imaginary nodes for order n), See Figure 1 for a complex variable element.
- Systemic conversion of the software to complex variable (first order) - or for higher order conversion to hypercomplex variable and overload all mathematical functions using a hypercomplex numerical library.
- Perturbation of the parameter(s) of interest along the imaginary axis(es):
    - o Shape sensitivities: use the imaginary nodal values to "perturb" the shape of the mesh per the sensitivity to compute.
    - o Material properties: perturb the material property of interest (imaginary nodal coordinates are zero).
    - o Forces: perturb the force along imaginary axes (imaginary nodal coordinates are zero).
- Formation of the complex (or hypercomplex) stiffness matrix and right hand side using the standard finite element formulation, $K = \int_{\Omega} B^{*T} D B^{*} |J^{*}| d\Omega$ where the "*" indicates a complex/hypercomplex

- Solution of a complex/hypercomplex system of equations to obtain sensitivities of the displacement with respect to the parameter(s) of interest.
- Post processing of the hypercomplex results to obtain sensitivities of strains, stresses, strain energy, etc. with respect to the parameter(s) of interest.
- The ERR is determined from the imaginary component of the strain energy divided by the step size used, e.g.

  $$U^* = \frac{1}{2}(u^*)^T K^* u^*$$

  and

  $$G = \frac{1}{h} Im(U^*)$$

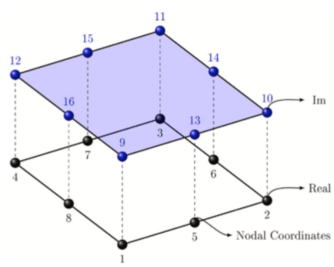

Fig. 1 Example of a 2D 8-noded complex element

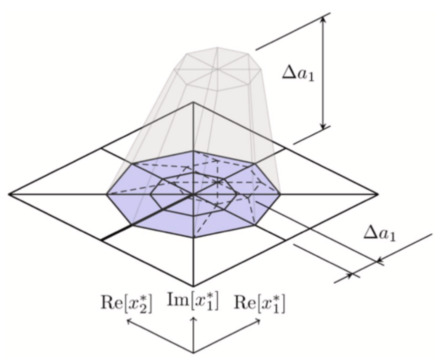

Fig. 2 Schematic of self-similar crack extension. Imaginary nodes are used to specify the perturbation of each real node.

2. Results

ZFEM analyses were performed using a CT specimen under 5 different loading levels. Figure 3 (left) show the FE mesh with plastic conditions shown in color. Figure 3 (right) shows the agreement between the ZFEM result (black dashed line) and the J integral results as a function of the number of contours. Similar accuracies are obtained for 3D analyses.

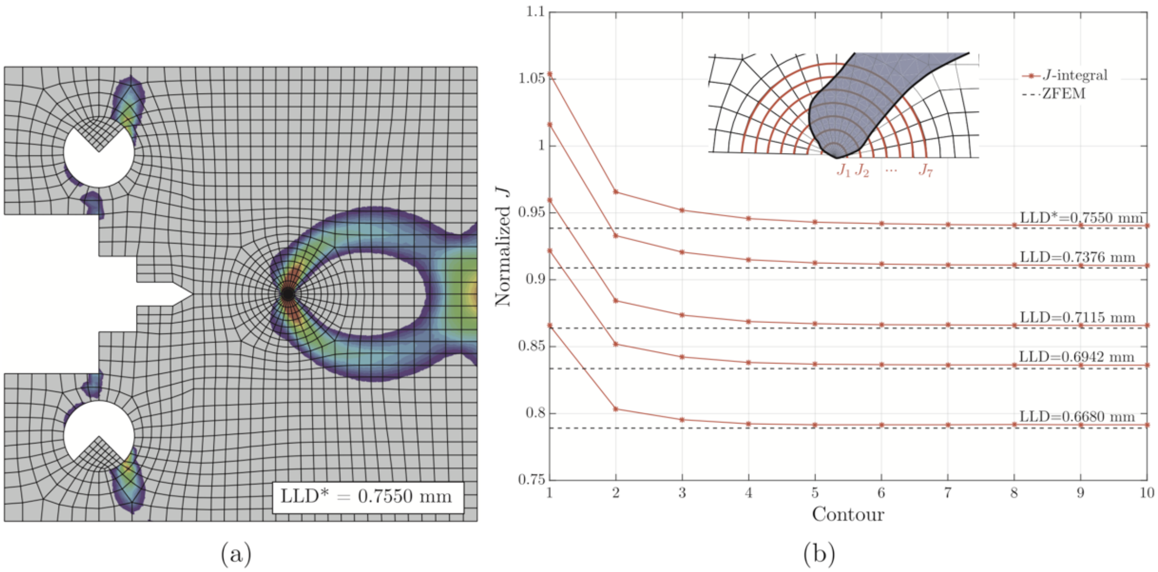

Fig. 3 a) CT FE specimen with plastic conditions in color, b) ERR results for ZFEM (dashed black line) and J integral contours.

3. Conclusions

The use of a complex variable finite element formulation for first order sensitivity analysis and a hypercomplex variable formulation for higher order sensitivities is an attractive method for computing the energy release rate. The method provides highly accurate results without the use of conservation integrals. The benefits are that a single ERR result is provided that is accurate for both linear and nonlinear analysis for both loading and unloading, and that no conservation integrals need to be developed and programmed for new physics, e.g., FGMs. In addition, sensitivities of the ERR with respect to other parameters such as geometric features and material properties can be computed as higher order derivatives.

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